Diversification and Small Bets in Scalable Environments

Navanit Arakeri

navanit@gmail.com

October 2007

This paper aims to show that payoffs in scalable environments grow in proportion to the number of investments and that small bets have the same payoff structure as larger bets. I conclude that if one accepts that we are in a scalable environment, diversification needs to be treated as an asset class, and that investment strategies that make a vary large number of small bets can realize large target returns.

I. Power-Laws

By scalable environment, I mean any environment that shows the following probability distribution of occurrences:

$$P(x) = Cx^{-\alpha}$$

Where 'α' is the tail exponent and P(x) represents the probability of occurrence of 'x'.

Most modern environments show this probability distribution, including the movies, arts, and stock markets.

Note that the probability of any sample X being greater than x is given by:

$$P(X>x) = \left(\frac{C}{\alpha-1}\right)x^{-\alpha+1}$$

Note that power-law distributions diverge as x -> 0, and are hence invalid models for x->0. In practice, power-laws are observed for samples above a certain 'crossover' point (= x_{min}.) . This crossover point is difficult to estimate.[1]

II. Diversification

Consider N securities worth $D each at time T = 0. An agent invests in one of each security, thus spending $ND in the portfolio. Consider a target wealth level $ηND, a return of 'η' times the initial investment. Let 'η_i' be the return from investment 'i'.

P(target wealth) = $P(\eta_1 D + \eta_2 D + \eta_3 D + ... + \eta_N D > \eta ND)$

P(target wealth) = $P((\eta_1 + \eta_2 + \eta_3 + ... + \eta_N)*D > (\eta N)*D)$

P(target wealth) = $P(\{\eta_i\} > \eta)$ (1)

where P(target wealth) is the probability of attaining the target wealth and $\{\eta_i\}$ is the mean value of the individual returns. Now the mean for a power-law distribution is given by:

$$\frac{C}{(2-\alpha)} x^{-\alpha+2}$$

Evaluated from x_{min} to ∞ . Note this diverges to infinity for $\alpha < 2$.

For $\alpha > 2$ we have:

$$\frac{(\alpha - 1)}{(\alpha - 2)} x$$

with $x = x_{min}$.

II (a). Case 1: α > 2

Consider equation (1) again:

P(target wealth) = P($\{\eta_i\} > \eta$). The mean value, $\{\eta_i\}$ is given by:

$$\frac{(\alpha - 1)}{(\alpha - 2)} \eta(\min)$$

P(target wealth) is:

$$P\left(\frac{(\alpha - 1)}{(\alpha - 2)} \eta(\min) > \eta\right)$$

or

$$P\left(\eta(\min) > \frac{(\alpha-2)}{(\alpha-1)} \eta\right)$$

η_{min} is the 'crossover' value (discussed in section I) at which the distribution begins to show power law properties. This value is relatively large (i.e. $\eta_{min} >> 0$), but fixed and invariant with sample size. This means we can find a large enough target η such that η_{min} is much smaller than η. Hence:

$$P\left(\eta(\min) > \frac{(\alpha-2)}{(\alpha-1)} \eta\right) = 0$$

For large values of target return η. And hence:

P(target wealth) = 0

So for $\alpha > 2$ we can say nothing about the effect of diversification. We *can* say that targeting large return sizes in this environment is unrealistic.

II (b). Case 2: $\alpha \leq 2$
With α smaller than or equal to 2, we have the mean $\{\eta_i\}$ diverging to infinity. In practice, the sample average shows enormous instability. As the number of samples approaches infinity, the sample average must approach infinity (the mean). It follows that as the number of samples increase, the sample average will increase ([1] explains this clearly).

$\{\eta_i\} \to \infty$ as $N \to \infty$

This means:

$P(\{\eta_i\} > \eta)$ approaches unity as the number of samples increases. Hence, from equation (1):

$P(\text{target wealth}) = P(\{\eta_i\} > \eta) \to 1$ for N increasing.

P(target wealth) increases and approaches unity as diversification is increased. This is true even for high target η. More on this later.

III. Estimating α
Estimating the value of α is fraught with error because of sampling issues. As α approaches 2, the number of samples required to 'home in' on the exponent increases dramatically. Nassim Taleb [2] and Benoit Mandelbrot [3] have previously argued that in practice, it is best to assume $\alpha \leq 2$. Recall that in section (II (a).) I argued that targeting a large return in an environment where $\alpha > 2$ is unrealistic. Therefore, the presence of enormous returns in an environment (like entrepreneurship;) hints at an $\alpha \leq 2$, though it does not follow mathematically.

From a risk management perspective, it is imperative that decisions be made assuming $\alpha \leq 2$.

IV. Environments with $\alpha \leq 2$
There are a certain class of instruments in finance that almost certainly show power-law returns with $\alpha \leq 2$. Certain derivatives, variance swaps, and exotic options depend on the *variance* of a set of securities that show power-law returns. If the set of underlying securities show power-law behavior with exponent 'α', then a portfolio of derivatives on top of these securities will show a power-law behavior with '$\alpha/2$', i.e half the exponent of the underlying securities. If the underlying securities have an exponent $\alpha = 3$, then a derivatives portfolio on top of these will have an exponent $\alpha = 1.5$ ($\alpha \leq 2$).

From section (II (b)) it follows that the probability of attaining a given return with a portfolio of these derivatives increases with increasing diversification. As a matter of fact, the target return can be enormous and sufficient diversification can ensure a high probability of success. Note that diversification can exist in time and space.

V. Small Bets vs. Big Bets
Note that the value of the security $D drops out of the equation in (1). This also follows from the classic power-law relationship:

$$\frac{(P(X>nx))}{(P(X>x))} = n^{-\alpha+1}$$

Note that the R.H.S is only a function of 'n' and not 'x'. This means the probability of a bet doubling in size (n = 2) is independent of the size of the bet (x).

For a given amount of capitalization, it makes no difference if we invest in one big bet or one small bet. However, small bets allow for much higher diversification given a fixed capital.

Conclusion
In this paper I argued that returns in power-law environments increase with the level of diversification. Put another way, the probability of hitting a target return increases with increasing diversification in investments. A portfolio of highly diversified small bets in a power-law environment with $\alpha \leq 2$ can realize monstrous target returns with sufficient diversification.

I also argued that the presence of enormous returns in an environment hints at $\alpha \leq 2$ properties and that certain classes of financial instruments are almost certainly in power-law environments with $\alpha \leq 2$.

References
[1] "Power laws, Pareto distributions, and Zipf's law," Newman, M. E. J.

[2] "The Black Swan," Nassim Nicholas Taleb, 2007

[3] "The behavior of certain speculative prices," Benoit B. Mandelbrot, 1963

Notes
1. Reference [1] is the best introductory article on power-laws.

2. The expected largest value of N samples from a power-law distribution is:

$$\{X\} = N^{\frac{1}{\alpha-1}}$$

For $\alpha > 1$, the largest value increases with N. [1].

3. Certain derivatives instruments can be structured to return payoffs as a function of the kurtosis (fourth moment) of the underlying instruments. These derivatives diverge much more quickly and hence smaller values of diversification may be sufficient for a given target return.

Thank you for reading. Questions and comments: navanit@gmail.com